Section 2.3: The Product and Quotient Rules and Higher Order Derivatives
Section 2.4: Chain Rule

Note: The derivative of a product is not the product of the derivatives.

Example 1: Differentiate the function $y = x^3 x^7$.

Solution:

■

The product rule is used to differentiate the product of two functions.

Product Rule

If $P(x) = f(x)\,g(x)$, then

$$P'(x) = f(x)\,g'(x) + g(x)\,f'(x)$$

$$\begin{matrix}\text{Derivative of}\\ \text{a Product}\end{matrix} = \begin{pmatrix}\text{First}\\ \text{Function}\end{pmatrix}\begin{pmatrix}\text{Derivative of the}\\ \text{Second Function}\end{pmatrix} + \begin{pmatrix}\text{Second}\\ \text{Function}\end{pmatrix}\begin{pmatrix}\text{Derivative of the}\\ \text{First Function}\end{pmatrix}$$

Example 2: Use the product rule to differentiate the function $y = x^3\, x^7$.

Solution:

$\blacksquare$

Example 3: Differentiate the function $y = (x^3 - 1)(x + 1)$.

Solution:

$\blacksquare$

Example 4: Differentiate the function $y = \sqrt{x}\, e^{x}$.

Solution:

∎

Example 5: Differentiate the function $y = x^{2} \sin x$.

Solution:

Quotient Rule

$$\text{If } Q(x) = \frac{f(x)}{g(x)}\text{, then}$$

$$Q'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

$$\begin{matrix}\text{Derivative of}\\ \text{a Quotient}\end{matrix} = \frac{(\text{Denominator})\begin{pmatrix}\text{Derivative of the}\\ \text{Numerator}\end{pmatrix} - (\text{Numerator})\begin{pmatrix}\text{Derivative of the}\\ \text{Denominator}\end{pmatrix}}{(\text{Denominator})^2}$$

Example 6: Differentiate the function $y = \dfrac{t^3 + t}{t^3 - 2}$.

Solution:

Example 7: Differentiate $y = \tan x$.

Solution:

$\blacksquare$

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \qquad\qquad \frac{d}{dx}(\csc x) =$$

$$\frac{d}{dx}(\sec x) = \qquad\qquad \frac{d}{dx}(\cot x) =$$

Example 8: Differentiate $g(t) = 4\sec t + \tan t$.

Solution:

$\blacksquare$

Example 9: Differentiate $y = \csc\theta(\theta + \cot\theta)$.

Solution: Since this function is the product of two functions of θ, we must use the product rule. This gives

$$\frac{dy}{d\theta} = \underbrace{\csc\theta}_{\text{First}} \underbrace{(1 + -\csc^2\theta)}_{\substack{\text{Derivative of} \\ \text{Second}}} + \underbrace{(\theta + \cot\theta)}_{\text{Second}} \underbrace{(-\csc\theta\cot\theta)}_{\substack{\text{Derivative of} \\ \text{First}}}$$

Distributing the trigonometric terms gives the result

$$\frac{dy}{d\theta} = \csc\theta - \csc^3\theta - \theta\csc\theta\cot\theta - \csc\theta\cot^2\theta$$

$\blacksquare$

Example 10: Find the equation of the tangent line to the graph of $y = \dfrac{x}{x^2+1}$ at the point (3, 0.3).

Solution: To find the equation any line, including a tangent line, we need the slope and at least one point. The point (3, 0.3) is given. To find a formula for calculating the slope of the tangent line, we need to find the derivative of the function, which in this case is done using the quotient rule. We calculate the derivative and simplify as follows:

$$\frac{dy}{dx} = \frac{\overset{\text{Denom}}{(x^2+1)}\;\overset{\text{Derivative of Numer}}{(1)} - \overset{\text{Numer}}{(x)}\;\overset{\text{Derivative of Denom}}{(2x)}}{\underset{(\text{Denom})^2}{(x^2+1)^2}}$$

$$\frac{dy}{dx} = \frac{x^2+1-2x^2}{(x^2+1)^2} \qquad \text{(Multiply } x^2+1 \text{ and } 1,\ x \text{ and } 2x\text{)}$$

$$\frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2} \qquad \text{(Simplify)}$$

We now find the slope of the tangent line by substituting the x coordinate of the point (3, 0.3) into the derivative. This gives

$$\underset{x=3}{\text{Slope of tangent line at the point } (3, 0.3)} = m = \frac{1-(3)^2}{(3^2+1)^2} = \frac{1-9}{(9+1)^2} = \frac{-8}{(10)^2} = -\frac{8}{100} = -0.08.$$

Using $m = -0.08$ and the point (3, 0.3), we solve of the parameter b in the equation of the tangent line as follows:

$$\begin{aligned}
&y = mx + b \\
&y = -0.08x + b \\
&0.3 = -0.08(3) + b && \text{(For the point } (3, 0.3),\ x = 3 \text{ when } y = 0.3\text{)} \\
&0.3 = -0.24 + b && \text{(Simplify_} \\
&b = 0.3 + 0.24 = 0.54 && \text{(Solve for } b\text{)}
\end{aligned}$$

Hence, the equation of the tangent line is $\boxed{y = -0.08x + 0.54}$.

■

The Second Derivative

Since the derivative is itself a function, we can calculate its derivative. The derivative of the first derivative is known as the second derivative.

Notations for the Second Derivative

	Prime Notations	$\frac{d}{dx}$ **Notations**
First Derivative Notations	$f'(x)$	$\frac{d}{dx}(f(x))$
	y'	$\frac{dy}{dx}$
Second Derivative Notations	$f''(x)$	$\frac{d}{dx}(\frac{d}{dx}(f(x))) = \frac{d^2}{dx^2}(f(x))$
	y''	$\frac{d^2y}{dx^2}$

Example 11: Find the second derivative for the function $f(x) = 2x^2 - x + 2e^x - 1$.

Solution:

Example 12: Find the second derivative for the function $y = \sqrt{x} - \dfrac{1}{x}$.

Solution:

$\blacksquare$

Velocity and Acceleration

Given a position function $y = s(t)$

Velocity: $v(t) = s'(t)$

Acceleration: Is the rate of change of velocity and is defined to be

$$a(t) = v'(t) = s''(t)$$

Example 13: Suppose the equation of motion of a particle is given by the position function $s(t) = 2t^3 - 7t^2 + 4t + 1$ where *s* is in meters and *t* is in seconds.

a. Find the velocity and acceleration functions as functions of *t*.
b. Find the acceleration after 1 second.

Solution:

$\blacksquare$